Lax-Wendroff Convergence Theorems for a Class of Overset Mesh Methods Based on Higher Order Flux Reconstruction

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Conservation Law Based Update Schemes

- In this talk convergence to a weak solution under grid refinement is demonstrated for a class of overset grid schemes based on higher order Flux Reconstruction (FR) as well as MUSCL schemes
- The method is based on enforcing the conservation laws on a cell-wise basis rather than the traditional approaches involving interpolation
- Sources of conservation error are clearly identified
- Consider the following conservation law

$$\frac{\partial q}{\partial t}+\frac{\partial f(q)}{\partial x}=0$$

Conservation Law Based Update Schemes

- In order to solve this equation numerically on a single grid, the domain is partitioned into discrete cells $j = \left[x_j - \frac{\Delta x}{2}, x_j + \frac{\Delta x}{2} \right]$

- The variable q is averaged over cell j as follows

$$q_j^n = \frac{1}{\Delta x} \int_{x_j - \frac{\Delta x}{2}}^{x_j + \frac{\Delta x}{2}} q(x, t_n)\,dx \;\; ; t_n = \mathrm{n}\,\Delta \mathrm{t}\,; \mathrm{etc.}$$

- The value of the cell averaged variable q_j in cell Ω_j is updated as follows

$$q_j^{n+1} = q_j^n - \frac{\Delta t}{\Delta x}\left(F\left(ql_{j+1}, qr_{j+1}\right) - F\left(ql_j, qr_j\right)\right)$$

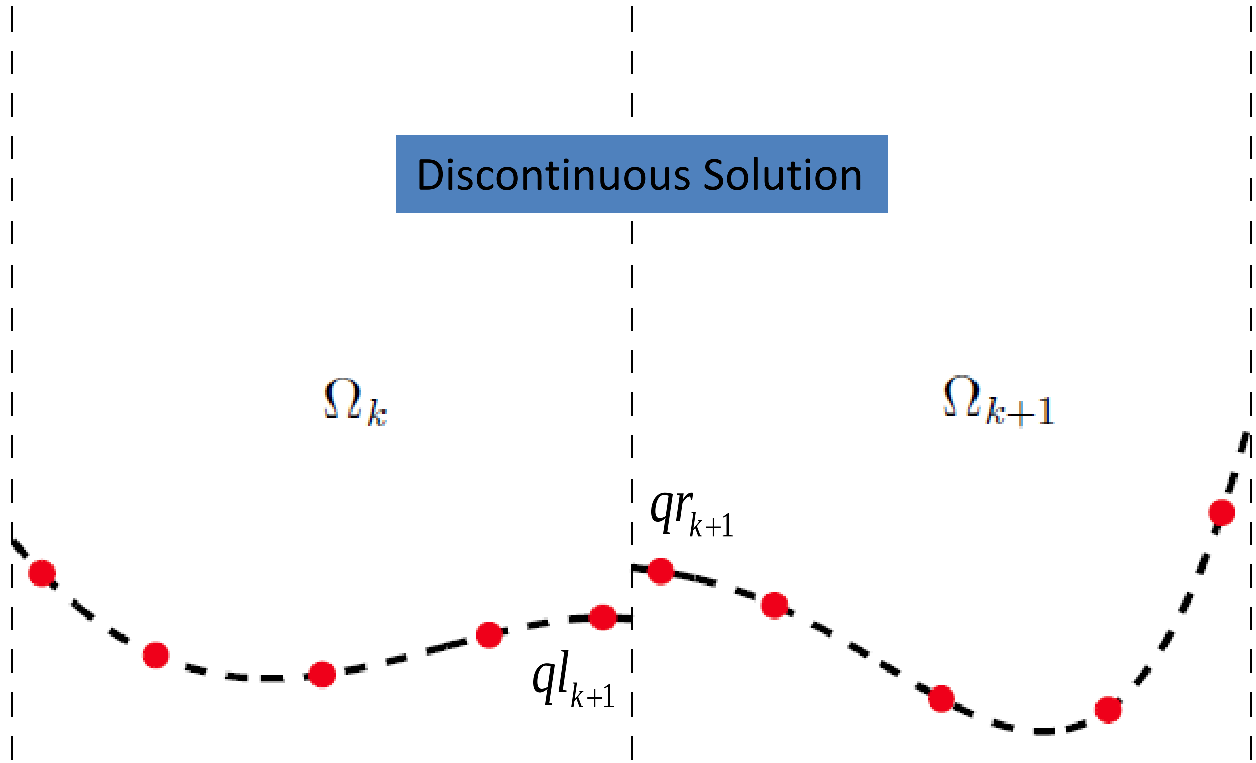
MUSCL Reconstruction

- Here ql, qr are the values of q reconstructed from the cell averages q_j^n and evaluated at the cell interfaces using the MUSCL procedure and F is a numerical flux function (Roe, Lax–Friedrichs, etc.)

a. Cell averaging of quartic data b. Linear reconstruction c. Quadratic reconstruction **(Barth and Ohlberger, 2004)**

- Extension to higher order in multiple dimensions requires large stencil sizes. This is impractical for overset grids.

Flux Reconstruction[1,2]

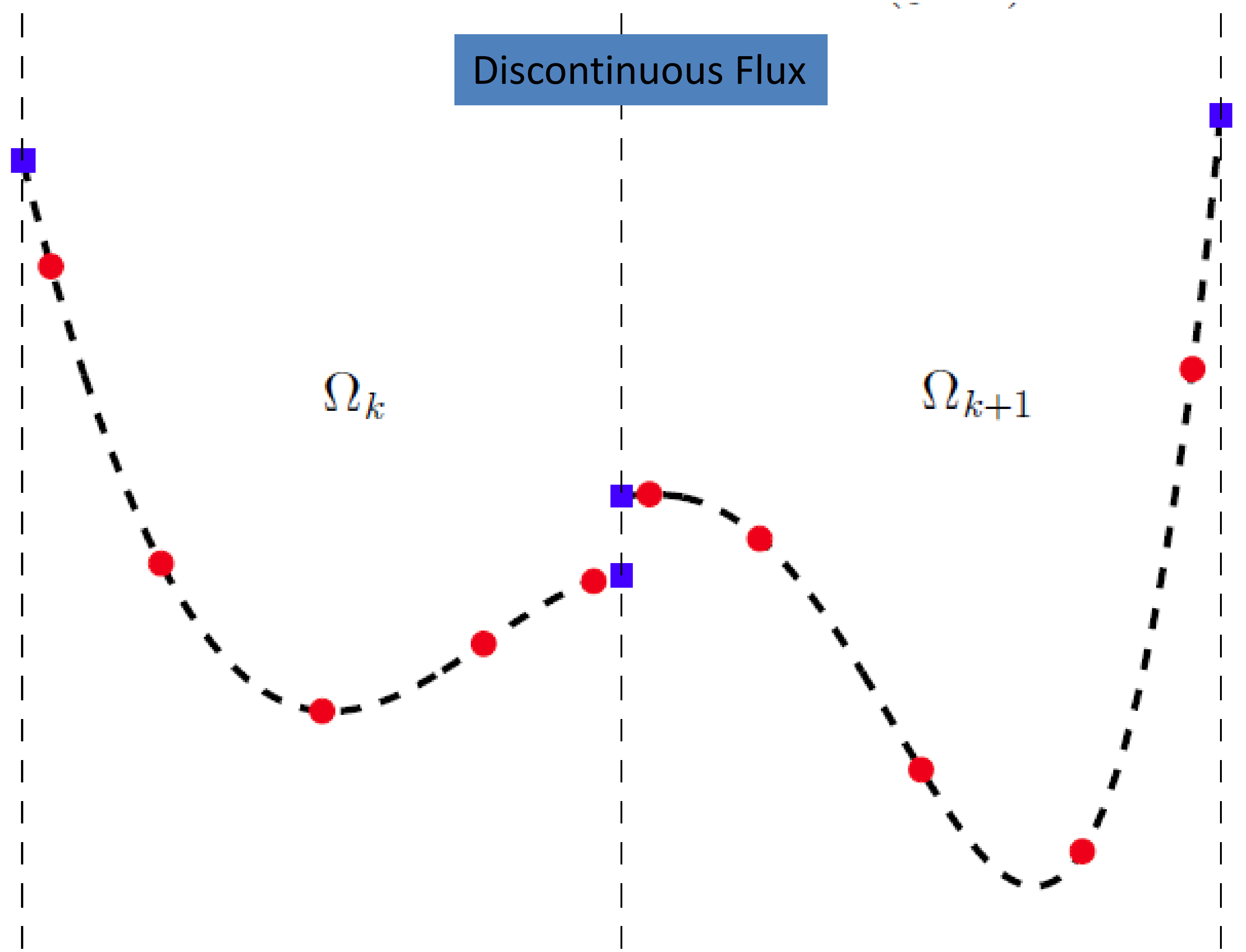


- The solution in each cell is approximated by *K* pieces of data
- A K-1 order polynomial is used to fit the data.
- ql_{k+1} and qr_{k+1} are defined by extrapolating the polynomials to the cell faces

1. Huynh, H. T. "A flux reconstruction approach to high-order schemes including discontinuous Galerkin methods." *AIAA paper* 4079 (2007): 2007.
2. A Novel Approach for Shock Capturing in Unstructured High-Order Methods, by Abhishek Sheshadri, Stanford University

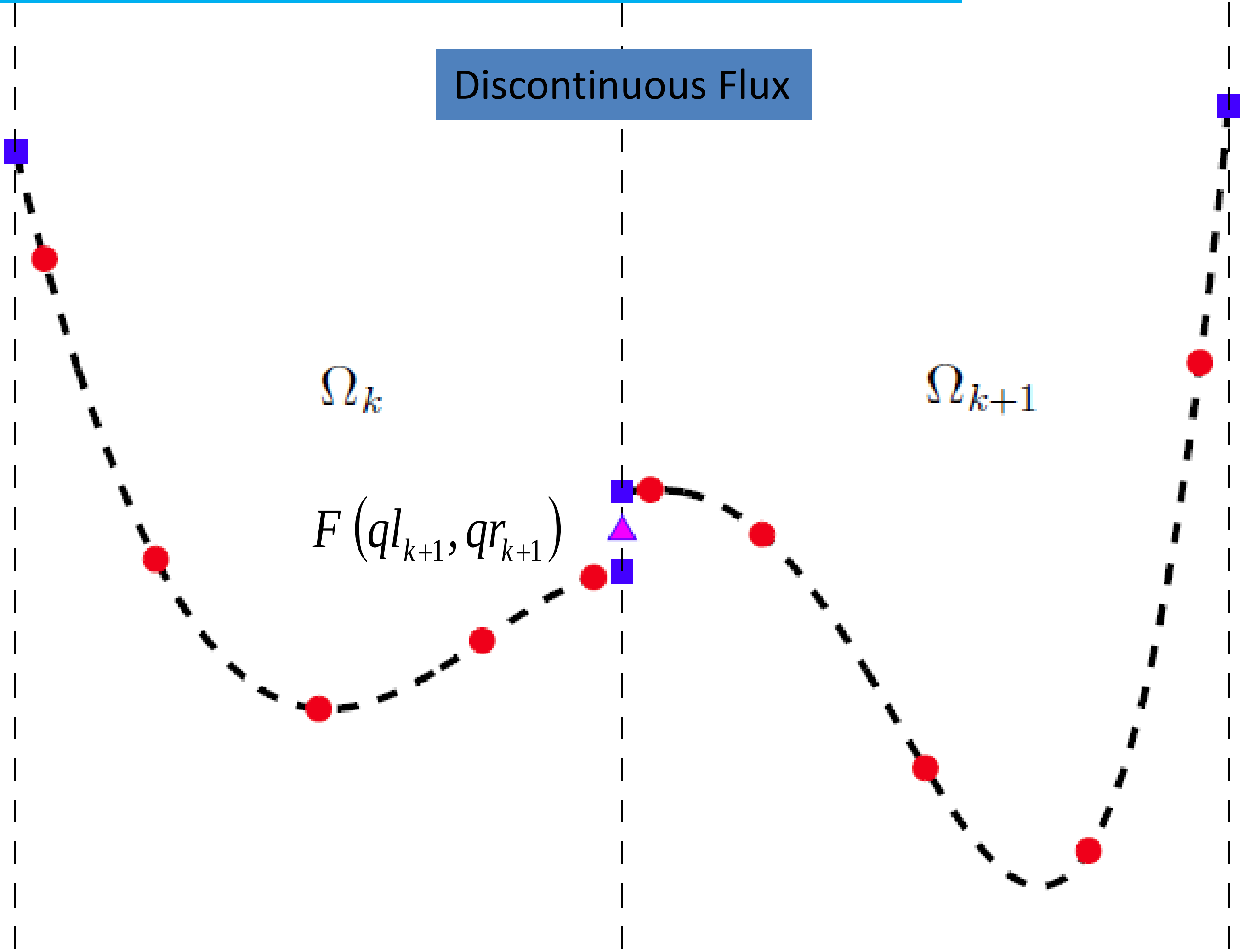
Flux Reconstruction

- A K-1 order polynomial is used to approximate the flux function



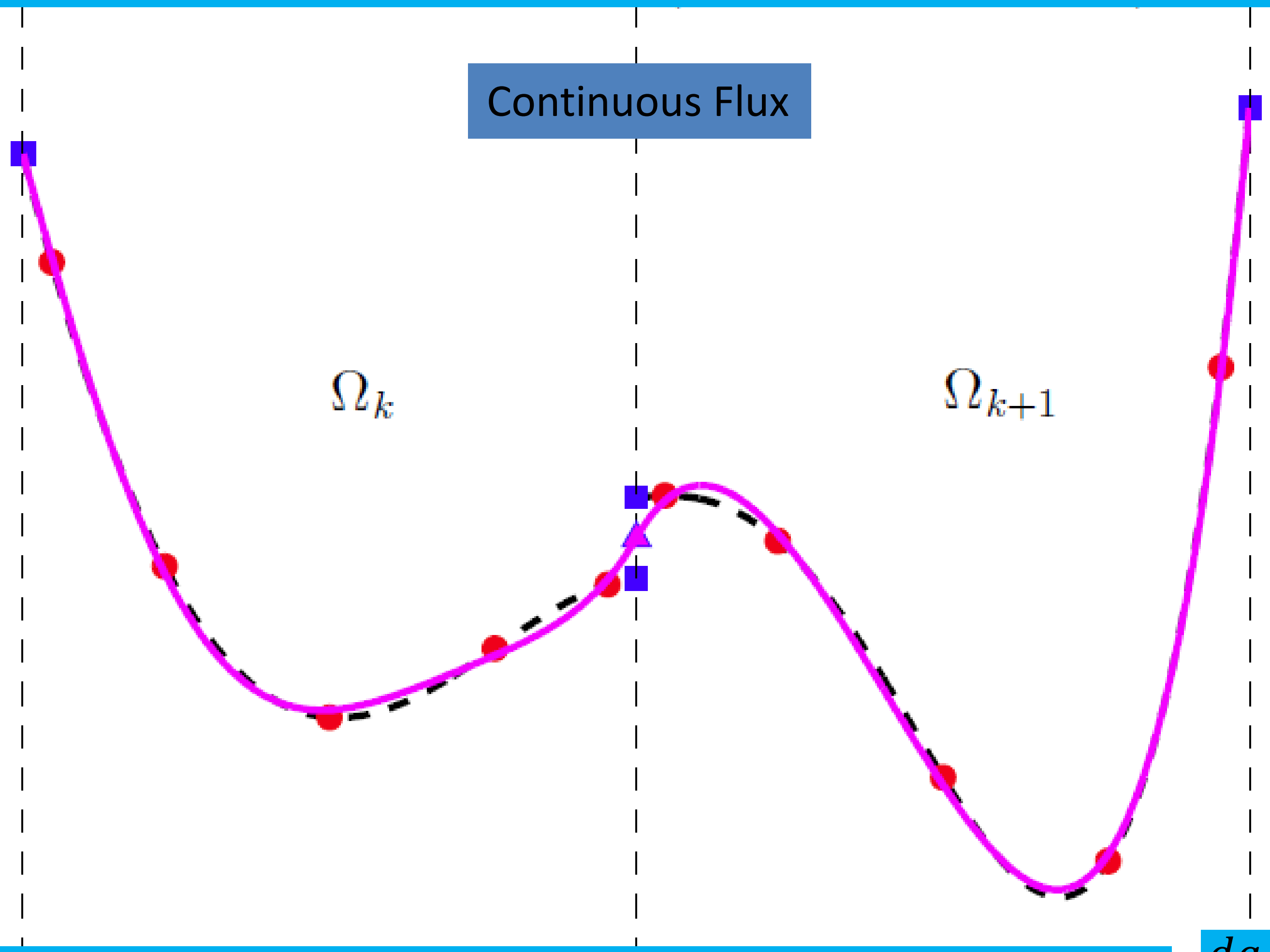
Flux Reconstruction

- ql_{k+1} and qr_{k+1} are used to define an upwind flux at the face



Flux Reconstruction

- Correction polynomials are used to create a continuous Kth order flux approximation F



- The continuous flux polynomial is used to time advance the variables

$$\frac{dq_{j,k}}{dt} = [F_x]_{j,k}$$

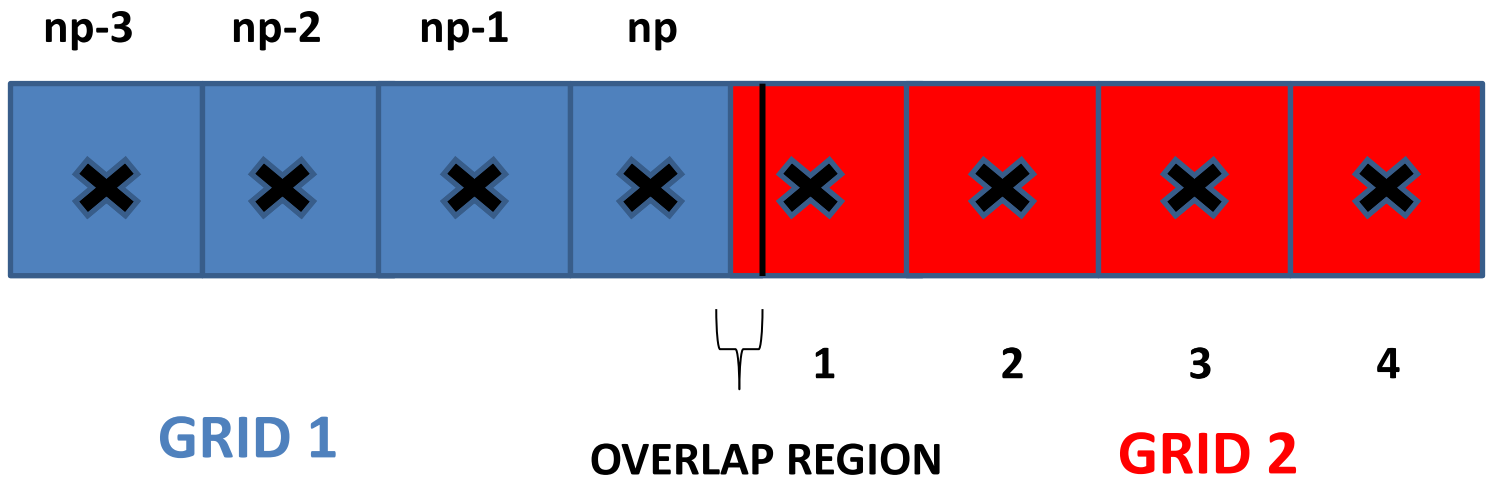
Conservation Law Based Update Schemes

• What is the "best" way to extend these approaches to cases of multiple overlapping grids?

•Rather than interpolate variables or fluxes seek alternate formulation

•Use data from both grids to reconstruct interface states

•Consider two grids with some small amount of overlap

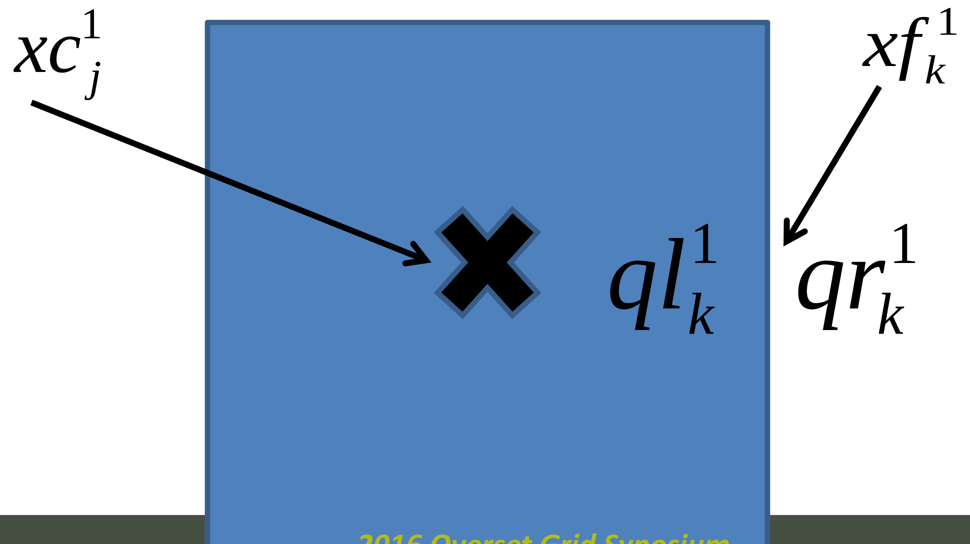


Conservation Law Based Update Schemes

•Let the value of the cell averaged variable in cell *j* of grid 1 be denoted as q_j^1, and the cell centroid value of cell j be denoted as xc_j^1 etc

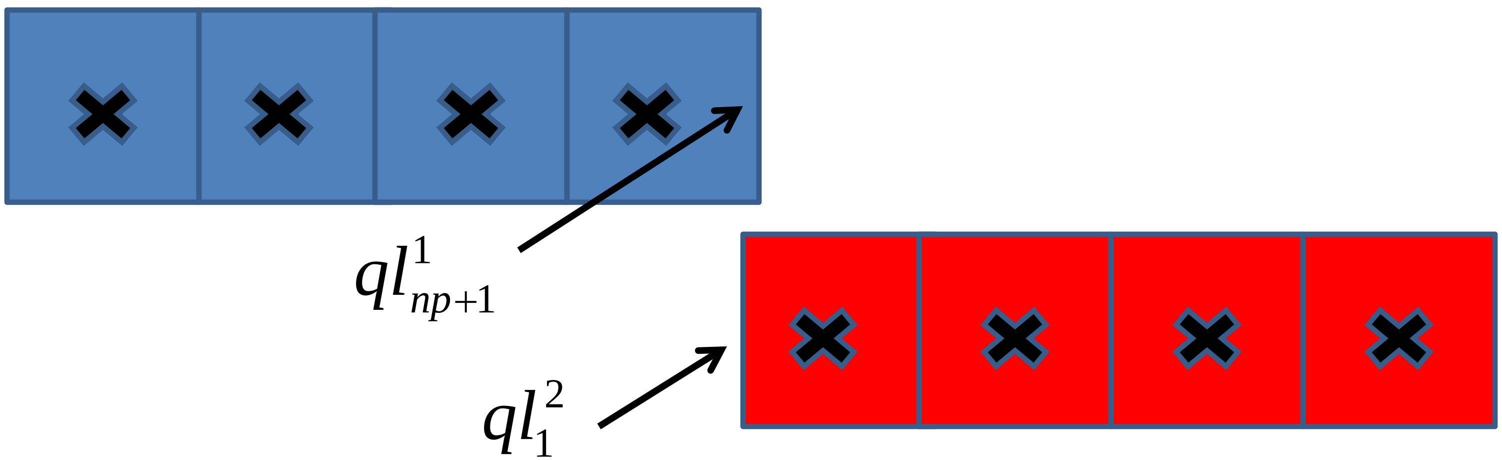
•Let the left and right interface states at face *k* of grid 1 be denoted as ql_k^1 and qr_k^1 respectively

•Let the face centroid of face *k* of grid 1 be denoted as xf_k^1 respectively

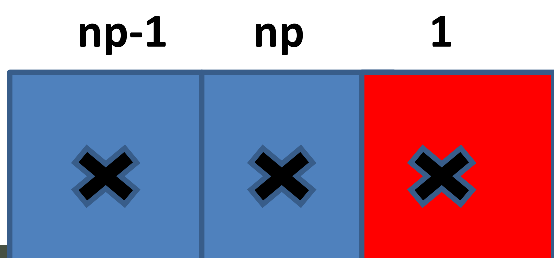


Conservation Law Based Update Schemes

- Consider computing the cell interface states ql^1_{np+1} and ql^2_1 as follows



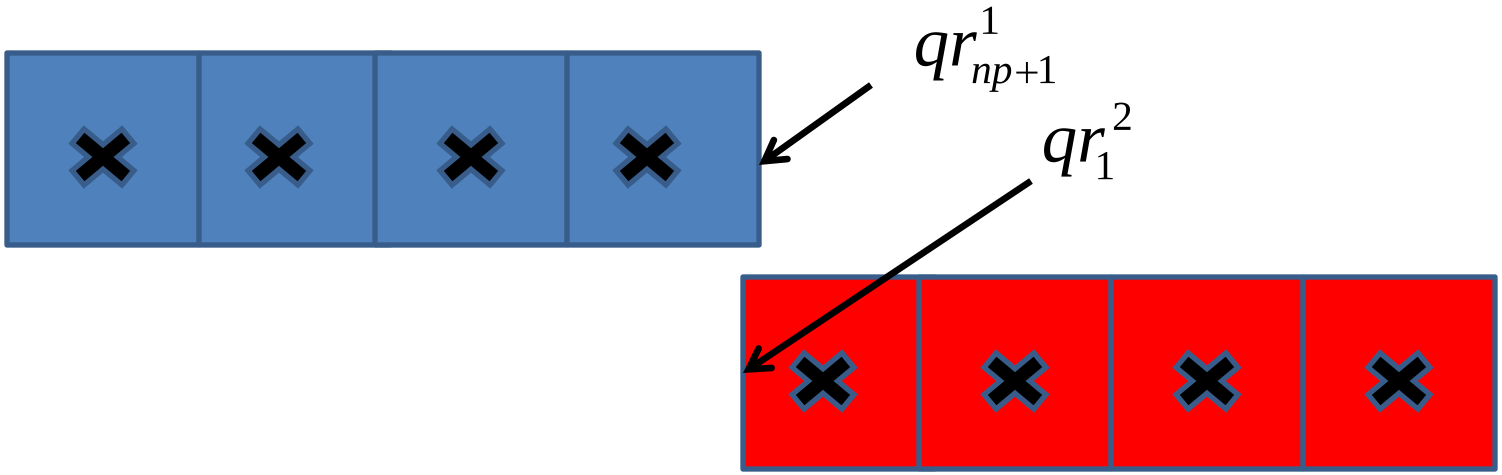
- The states are computed by constructing a least squares monotonically limited approximation to the solution gradient in cell np from mesh 1 using the cells below and then extrapolating q to the cell faces using this gradient



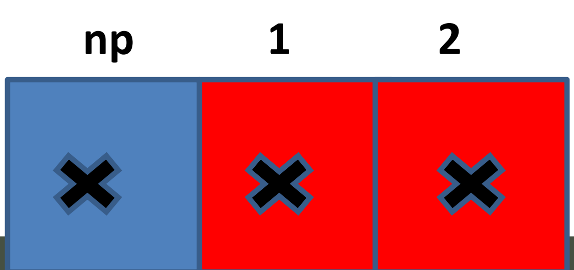
$$
\begin{aligned}
ql^1_{np+1} &= q^1_{np} + \overline{\nabla} q^1_{np} \bullet \left(xf^{\,1}_{np+1} - xc^1_{np} \right) \\
ql^2_1 &= q^1_{np} + \overline{\nabla} q^1_{np} \bullet \left(xf^{\,2}_1 - xc^1_{np} \right)
\end{aligned}
$$

Conservation Law Based Update Schemes

- Consider computing the cell interface states qr_{np+1}^{1} and qr_{1}^{2} as follows

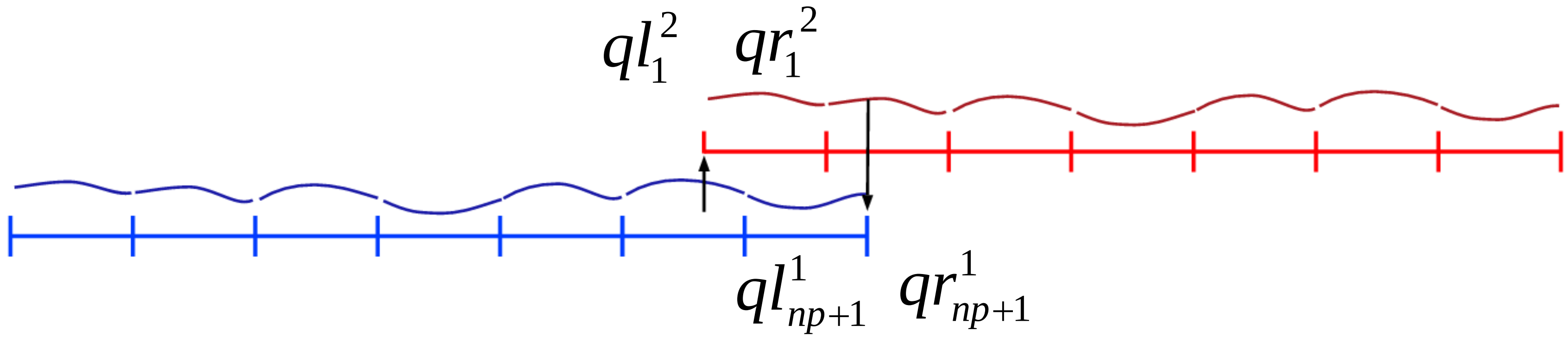


- The states are computed by constructing a least squares monotonically limited approximation to the solution gradient in cell 1 from mesh 2 using the cells below and then extrapolating q to the cell faces using this gradient



$$
\begin{aligned}
qr_{np+1}^{1} &= q_1^2 + \overline{\nabla} q_1^2 \bullet \left(xf_{np+1}^{1} - xc_1^2 \right) \\
qr_1^2 &= q_1^2 + \overline{\nabla} q_1^2 \bullet \left(xf_1^{2} - xc_1^2 \right)
\end{aligned}
$$

Data Exchange for Flux Reconstruction



1. Extrapolate ql_{np+1}^1 from boundary cell solution polynomial
2. Interpolate qr_{np+1}^1 from donor cell solution polynomial
3. Compute upwind flux $F\left(ql_{np+1}^1, qr_{np+1}^1\right)$

1. Galbraith, M. C., A Discontinuous Galerkin Overset Solver, Ph.D. thesis, University of Cincinnati, 2013.
2. Crabill, J., Jameson, A. and Sitaraman, J.: A High-Order Overset Method on Moving and Deforming Grids. AIAA 2016-3225, AIAA Aviation, AIAA Modeling and Simulation Technologies Conference, 13-17 June 2016, Washington, DC.

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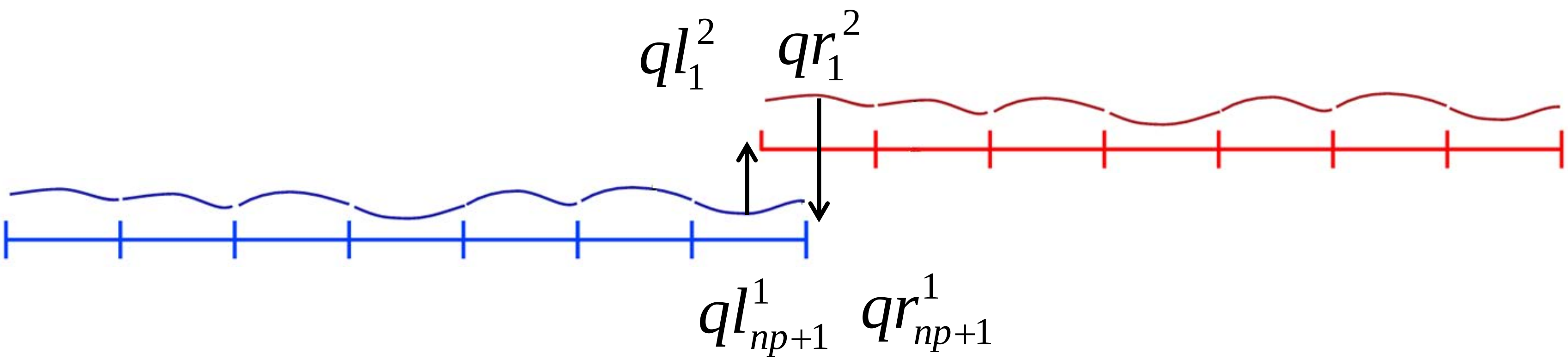

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Conservation Law Based Update Schemes

•A consistent and convergent scheme should be achieved as grid refinement is performed if the distance between the cells centers used in the reconstructions approach zero under grid refinement, i.e.

$$ql^{1}_{np+1} \to ql^{2}_{1} \text{ and } qr^{1}_{np+1} \to qr^{2}_{1} \text{ as } \Delta x \to 0$$

in that

$$\left\| qr^{1}_{np+1} - qr^{2}_{1} \right\| \cong \left\| \overline{\nabla} q^{2}_{1} \bullet \left(xf^{1}_{np+1} - xf^{2}_{1} \right) \right\| \leq \left\| \overline{\nabla} q^{2}_{1} \right\| \left\| \left(xf^{1}_{np+1} - xf^{2}_{1} \right) \right\| \leq TV(q)\Delta x$$

etc.

•Here TV is the total variation

•A similar argument holds for meshes with small gaps between the grids

•The basic argument remains unchanged in the case of general overlap

Generalized Lax-Wendroff Theorem

- Recall that a function $q(x,t)$ is considered to be a weak solution to the conservation law

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0$$

If for any compact, differentiable function $\phi(x,t)$ the following integrals are satisfied[1]

$$\int_{0}^{\infty} \int_{-\infty}^{\infty} \left[\phi_t q + \phi_x f(q)\right] dx\, dt = -\int_{-\infty}^{\infty} \phi(x,0) q(x,0)\, dx$$

1. LeVeque, Randall J., and Randall J. Leveque. *Numerical methods for conservation laws*. Vol. 132. Basel: Birkhäuser, 1992.

Generalized Lax-Wendroff Theorem

- In order to demonstrate that these integrals are satisfied for the proposed method, first multiply the cell average update equation by $\phi\left(x_j^1, t_n\right)$ for mesh 1, etc.

$$\phi(x_j^1,t_n)\left(q_j^{1,n+1} - q_j^{1,n}\right) = -\frac{\Delta t}{\Delta x}\phi(x_j^1,t_n)\left(F\left(ql_{j+1}^{1}, qr_{j+1}^{1}\right) - F\left(ql_j^{1}, qr_j^{1}\right)\right)$$

- Next sum over *j* and *n* for each mesh as follows

$$\Delta t \Delta x \left\{ \sum_{j=-\infty}^{np} \sum_{n=0}^{\infty} \phi(x_j^1,t_n)\frac{\left(q_j^{1,n+1} - q_j^{1,n}\right)}{\Delta t} + \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \phi(x_j^2,t_n)\frac{\left(q_j^{2,n+1} - q_j^{2,n}\right)}{\Delta t} = \right.$$

$$-\sum_{j=-\infty}^{np} \sum_{n=0}^{\infty} \phi(x_j^1,t_n)\frac{\left(F\left(ql_{j+1}^{1}, qr_{j+1}^{1}\right) - F\left(ql_j^{1}, qr_j^{1}\right)\right)}{\Delta x}$$

$$\left. -\sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \phi(x_j^2,t_n)\frac{\left(F\left(ql_{j+1}^{2}, qr_{j+1}^{2}\right) - F\left(ql_j^{2}, qr_j^{2}\right)\right)}{\Delta x} \right\}$$

Generalized Lax-Wendroff Theorem

- The following summation by parts formula is useful for the proving convergence of the proposed scheme to a weak solution

$$\sum_{j=0}^{N} a_j \left(b_{j+1} - b_j\right) = a_N b_{N+1} - a_0 b_0 - \sum_{j=1}^{N} b_j \left(a_j - a_{j-1}\right) \quad \text{SPB}$$

Generalized Lax-Wendroff Theorem

- For the LHS use the SPB formula on the n sum

$$
-\sum_{j=-\infty}^{np} \Delta x\, \phi(x_j^1, t_0) q_j^{1,0} - \sum_{j=1}^{\infty} \Delta x\, \phi(x_j^2, t_0) q_j^{2,0}
$$

$$
-\sum_{n=1}^{\infty} \Delta t \left(\sum_{j=-\infty}^{np} \Delta x\, q_j^{1,n} \left(\frac{\phi(x_j^1, t_n) - \phi(x_j^1, t_{n-1})}{\Delta t} \right) + \sum_{j=1}^{\infty} \Delta x\, q_j^{2,n} \left(\frac{\phi(x_j^2, t_n) - \phi(x_j^2, t_{n-1})}{\Delta t} \right) \right)
$$

- Note that these terms are discrete approximations to the following integrals

$$
-\int_{-\infty}^{\mathrm{xf}_{\mathrm{np}+1}^{1}} \phi(x,0) q(x,0)\, dx - \int_{\mathrm{xf}_1^2}^{\infty} \phi(x,0) q(x,0)\, dx - \int_0^{\infty} \left(\int_{-\infty}^{\mathrm{xf}_{\mathrm{np}+1}^{1}} \phi_t q + \int_{\mathrm{xf}_1^2}^{\infty} \phi_t q \right) dx\, dt
$$

Generalized Lax-Wendroff Theorem

- For the RHS use the SPB formula on the j sum

$$\sum_{n=0}^{\infty}\Delta t\left(\sum_{j=-\infty}^{np}\Delta x\, F\left(ql_j^{1},qr_j^{1}\right)\left(\frac{\phi(x_j^1,t_n)-\phi(x_{j-1}^1,t_n)}{\Delta x}\right)+\sum_{j=2}^{\infty}\Delta x\, F\left(ql_j^{2},qr_j^{2}\right)\left(\frac{\phi(x_j^2,t_n)-\phi(x_{j-1}^2,t_n)}{\Delta x}\right)\right)$$

$$-\sum_{n=0}^{\infty}\Delta t\Delta x\left(F\left(ql_{np+1}^{1},qr_{np+1}^{1}\right)\frac{\phi(x_{np}^1,t_n)}{\Delta x}-F\left(ql_1^{2},qr_1^{2}\right)\frac{\phi(x_1^2,t_n)}{\Delta x}\right)$$

End Terms from SBP Formula

Generalized Lax-Wendroff Theorem

- The end terms at the grid interface become

$$-\sum_{n=0}^{\infty} \Delta t \Delta x \left(F\left(ql_{np+1}^{1}, qr_{np+1}^{1}\right) \frac{\phi(x_{np}^{1}, t_n)}{\Delta x} - F\left(ql_1^{2}, qr_1^{2}\right) \frac{\phi(x_1^{2}, t_n)}{\Delta x} \right) =$$

$$\sum_{n=0}^{\infty} \Delta t \, \Delta x \left(F\left(ql_1^{2}, qr_1^{2}\right) \left(\frac{\phi(x_1^{2}, t_n) - \phi(x_{np}^{1}, t_n)}{\Delta x} \right) + \frac{\Delta F_{12} \phi(x_{np}^{1}, t_n)}{\Delta x} \right)$$

where $\Delta F_{12} = F\left(ql_1^{2}, qr_1^{2}\right) - F\left(ql_{np+1}^{1}, qr_{np+1}^{1}\right)$

Generalized Lax-Wendroff Theorem

- We can show that $\Delta F_{12} \to 0$ as $\Delta x \to 0$ as follows

$$\Delta F_{12} = F\left(ql_1^{\,2}, qr_1^{\,2}\right) - F\left(ql_{np+1}^{\,1}, qr_{np+1}^{\,1}\right)$$

$$\Delta F_{12} = F\left(ql_1^{\,2}, qr_1^{\,2}\right) - F\left(ql_1^{\,2} + (ql_{np+1}^{\,1} - ql_1^{\,2}), qr_1^{\,2} + (qr_{np+1}^{\,1} - qr_1^{\,2})\right)$$

$$\left\|\Delta F_{12}\right\| \le C \max(\left\|ql_{np+1}^{\,1} - ql_1^{\,2}\right\|, \left\|qr_{np+1}^{\,1} - qr_1^{\,2}\right\|)$$

F is Lipschitz Continuous

- Given that

$$\left\|qr_{np+1}^{\,1} - qr_1^{\,2}\right\| \cong \left\|\overline{\nabla} q_1^2 \bullet \left(xf_{np+1}^{\,1} - xf_1^{\,2}\right)\right\| \le \left\|\overline{\nabla} q_1^2\right\| \left\|\left(xf_{np+1}^{\,1} - xf_1^{\,2}\right)\right\| \le TV(q)\Delta x, \text{ etc.}$$

it follows that

$$\Rightarrow 0 \text{ as } \Delta x \Rightarrow 0$$

$$\left\|\Delta F_{12}\phi(x_{np}^1, t_n)\right\| \le C\, TV(q^n)\Delta x \left\|\phi(x_{np}^1, t_n)\right\|$$

Generalized Lax-Wendroff Theorem

- Note that these terms are discrete approximations to the following integrals

$$\int_{0}^{\infty}\int_{-\infty}^{xf_{np+1}^{1}} f(q)\phi_x\,dx\,dt + \int_{0}^{\infty}\int_{xf_1^{2}}^{\infty} f(q)\phi_x\,dx\,dt$$

- If $\left\| xf_{np+1}^{1} - xf_1^{2} \right\| \to 0$ as $\Delta x \to 0$ then terms like

$$-\int_{-\infty}^{xf_{np+1}^{1}} \phi(x,0)q(x,0)dx - \int_{xf_1^{2}}^{\infty} \phi(x,0)q(x,0)dx - \int_{0}^{\infty}\left(\int_{-\infty}^{xf_{np+1}^{1}} \phi_t q + \int_{xf_1^{2}}^{\infty} \phi_t q\right)dxdt$$

become $-\int_{-\infty}^{\infty} \phi(x,0)q(x,0)dx - \int_{0}^{\infty}\int_{-\infty}^{\infty} \phi_t q\,dx\,dt$

and the integral conservation law is satisfied under the same caveats invoked for the single grid case

Observations Based on Lax-Wendroff Theorem (1)

- If standard linear interpolation methods are used then no GENERAL bounds may be established a priori for the ΔF_{12} term

- We end up with a source term in the integral conservation law which MAY not vanish under grid refinement in the presence of discontinuous solutions.

$$\approx \int_0^\infty \Delta F_{12}(t)\varphi\left(x_R^1, t\right)dt$$

- These source terms become ODEs along characteristics and may globally pollute the solution
    - Corrupt convergence rates
    - Generate spurious waves

Observations Based on Lax-Wendroff Theorem (2)

- These terms can and will corrupt solutions when:
  - Severe mismatches in cell sizes exist
  - Strong discontinuities exist in the grid interface region
  - Slow moving discontinuities exist in the grid interface region
  - ***All three exist simultaneously***!

FR Shock Capturing Considerations

- Shock capturing schemes needed to control $TV(q)$

- Follow Zang and Shu.[1]

- Let $\{q\}_K^N$ be the solution points in cell Ω_N

- Use a discontinuity detector θ based on cell average values to control oscillations as follows: $\{q\}_K^N \rightarrow \theta(\{q\}_K^N - \bar{q}) + \bar{q}$
- Here we use Nichols discontinuity dector[2]
- One can also use the TVB filters of Engquist, et al[3]

$$\{q\}_K^N \rightarrow \theta(\{q\}_K^N - \{\tilde{q}\}_K^N) - \{\tilde{q}\}_K^N$$

1. Zhang, Xiangxiong, and Chi-Wang Shu. "On maximum-principle-satisfying high order schemes for scalar conservation laws." *Journal of Computational Physics* 229.9 (2010): 3091-3120.
2. Tramel, Robert W., Robert H. Nichols, and Pieter G. Buning. "Addition of improved shock-capturing schemes to OVERFLOW 2.1." *AIAA Paper* 3988 (2009): 2009.
3. Engquist, Björn, Per Lötstedt, and Björn Sjögreen. "Nonlinear filters for efficient shock computation." *Mathematics of Computation* 52.186 (1989): 509-537.

Sample Cases

- Shock Wave Formation in Burgers Equation

$$q(x,0) = \sin(x)$$

- Shock Propagation in Burgers Equation

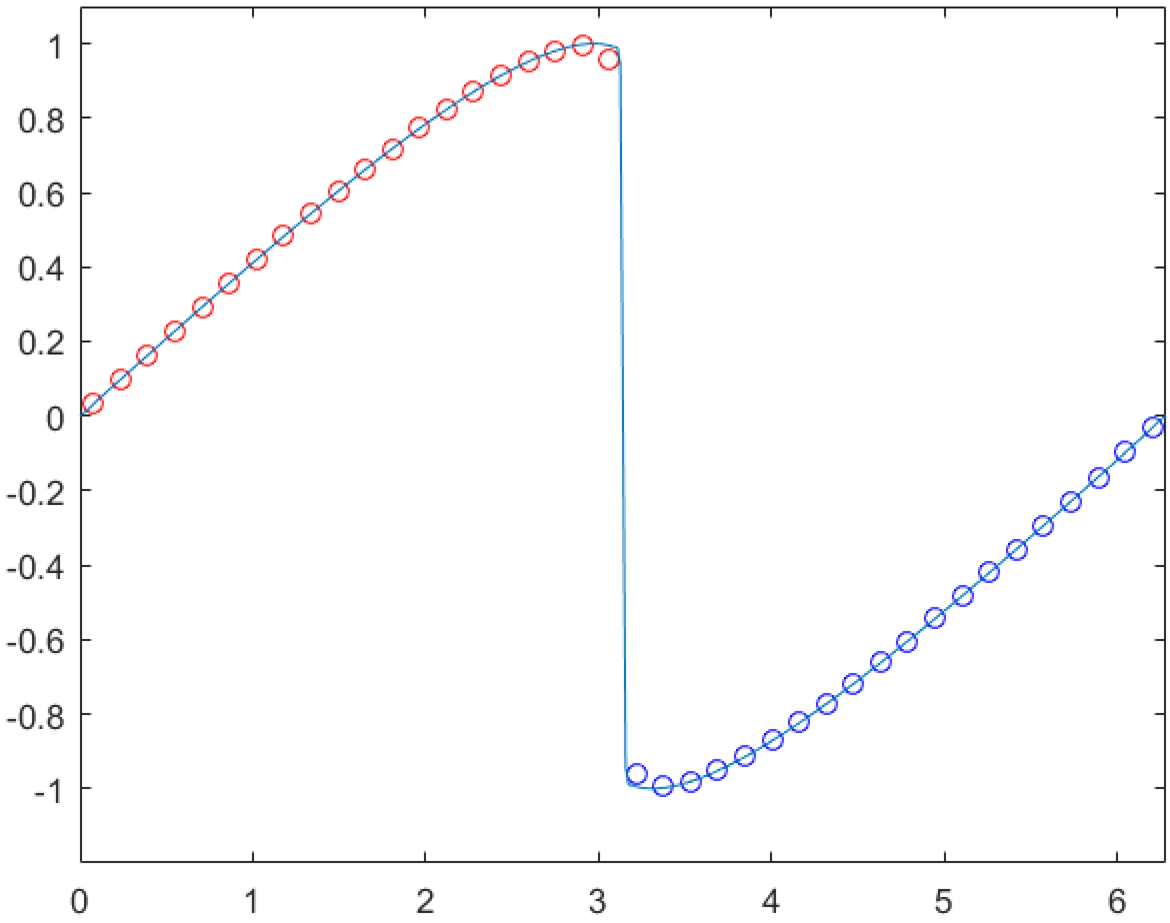
$$q(x,0) = \mathrm{H}(-(x - x_o))$$

- K=5
- Cells overlap 1/8 ΔX
- Shock location lies in the overset region
- Use global Lax-Friedrichs flux
- Use Gauss points and weights and Radau correction polynomials
- "Exact" solution for shock formation is a 4000 point solution using a 5th order WENO-RBF method

No Shock Treatment

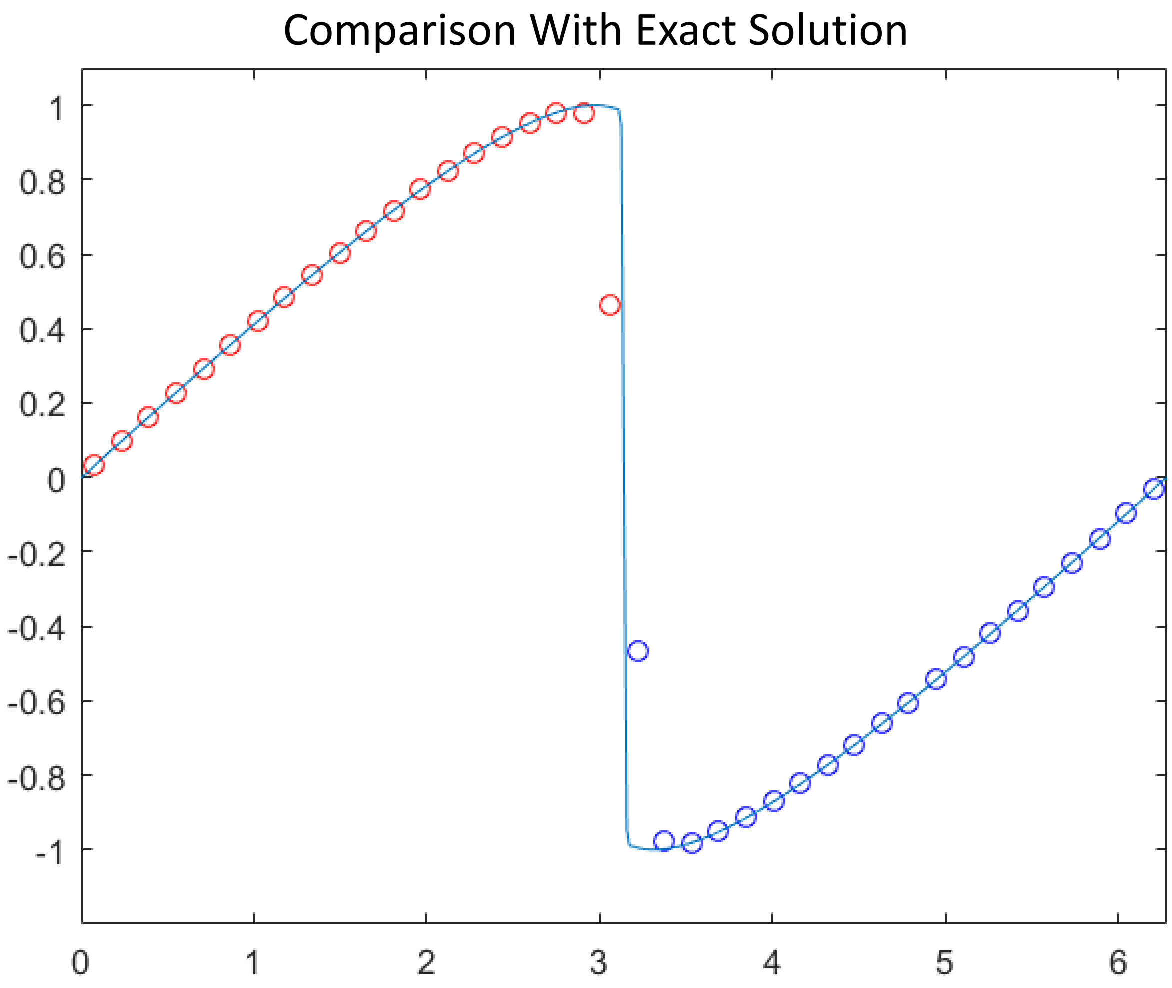
No Shock Treatment

Comparison With Exact Solution

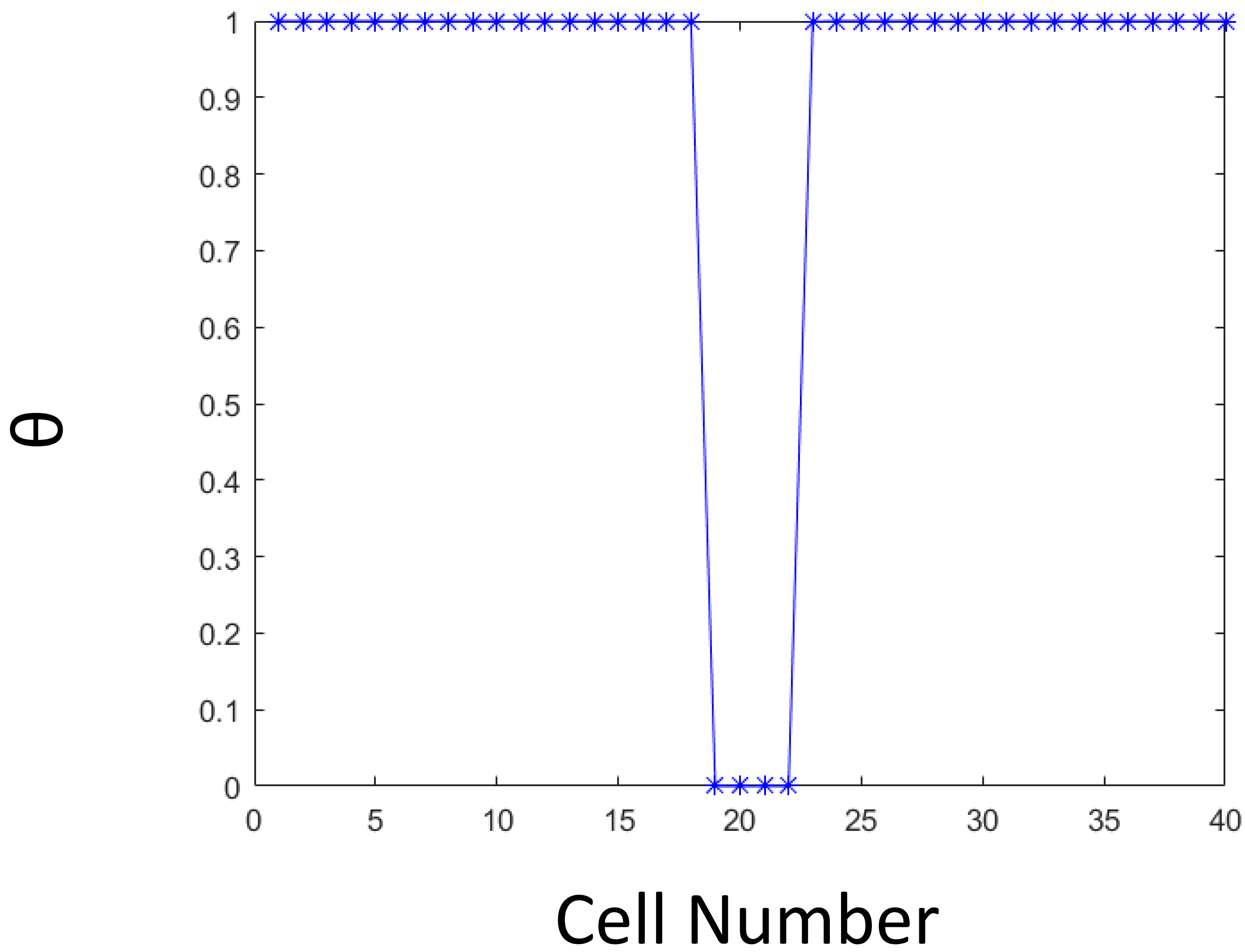


Dissipation Switch

Dissipation Switch



Discontinuity Detector



TVB Filter

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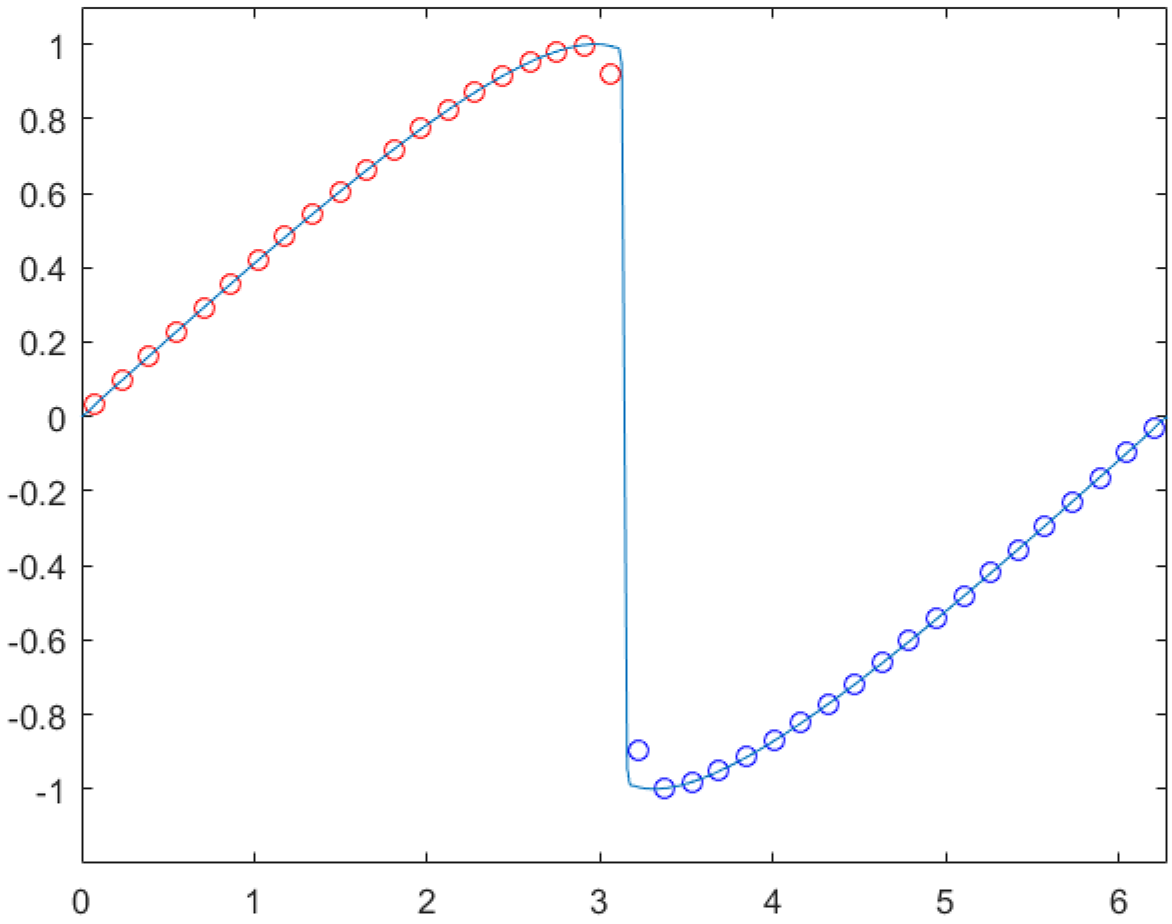
Comparison With Exact Solution



No Shock Treatment

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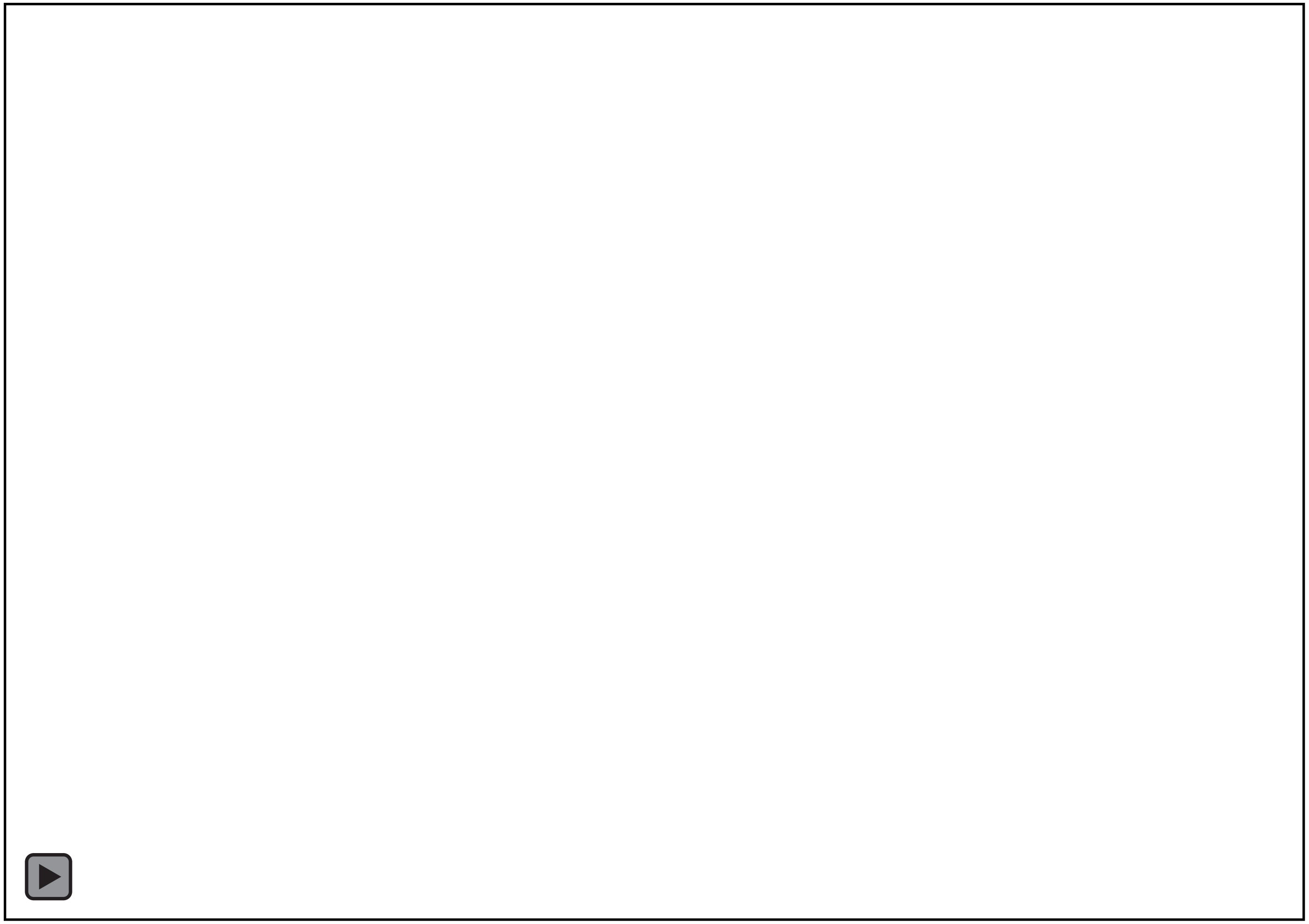

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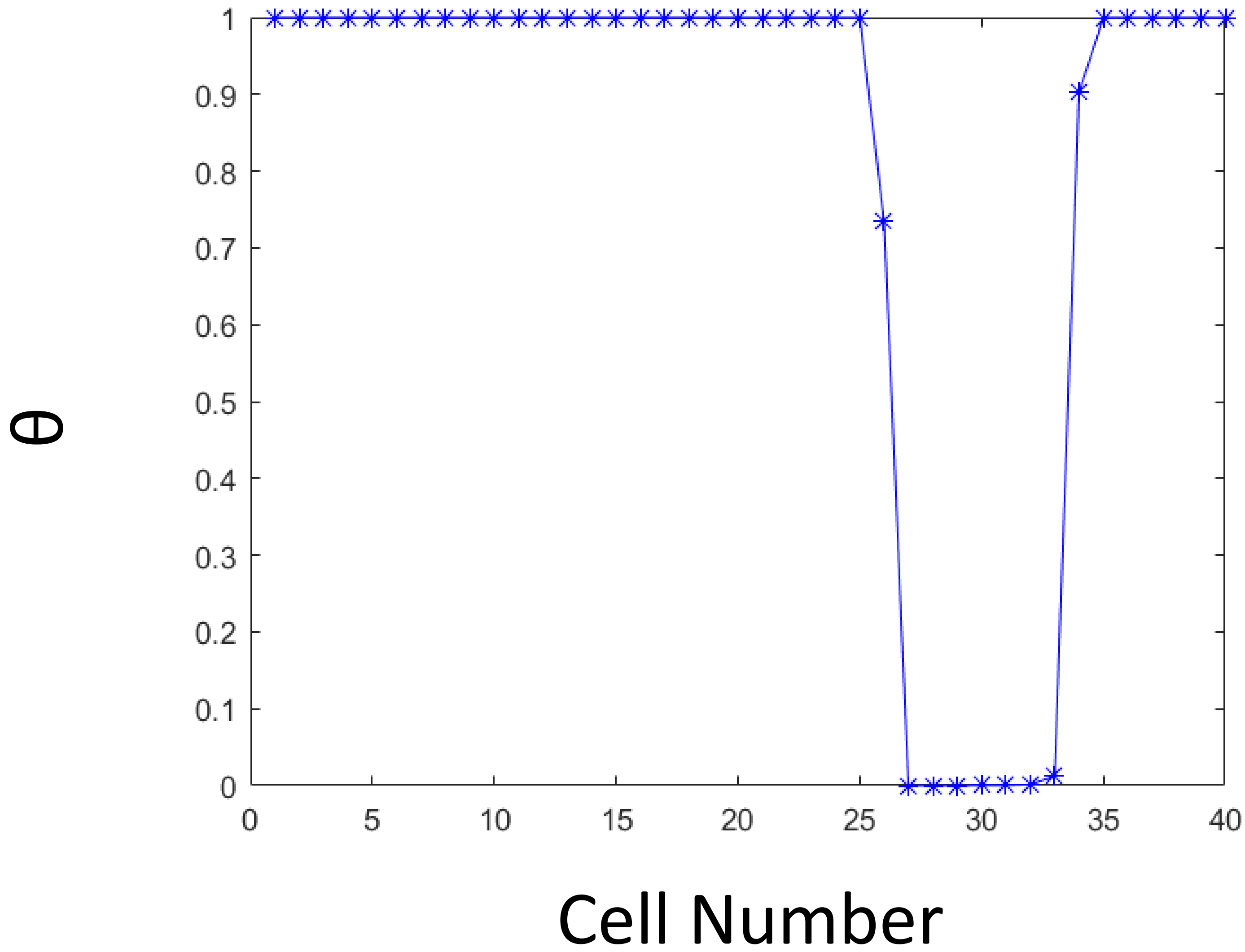

Conclusions

- Within the context of this framework strict lack of conservation becomes merely another source of numerical error to be removed by grid refinement

- Convergence to a weak solution is **guaranteed** as grid independence is achieved

- MUSCL/WENO/RBF schemes can be mixed with Flux Reconstruction schemes

- Framework seamlessly blends block matching/overset grid grids

ACKNOWLEDGEMENTS

- Thanks to
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for discussions and careful reading of earlier versions of this presentation

That's All Folks

- Questions
- Constructive Criticism

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